

Qualitative stability of nonlinear networked systems

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Abstract—In many large systems, such as those encountered in biology or economics, the dynamics are nonlinear and are only known very coarsely. It is often the case, however, that the signs (excitation or inhibition) of individual interactions are known. This paper extends to nonlinear systems the classical criteria of linear sign stability introduced in the 70's, yielding simple sufficient conditions to determine stability using only the sign patterns of the interactions.

Index Terms—nonlinear system, networks, stability.

I. INTRODUCTION

It is challenging to determine the stability of large complex systems—such as gene regulation networks, ecological systems and economic markets—not only because of their size, but because in practice we only know their dynamics very coarsely. Our knowledge of these large complex systems is often limited to the interaction network between the system's components (nodes) and the sign-pattern of the network [1]. In gene regulation systems, for instance, we know the complete list of genes—the nodes of the network—and the signs of their interactions—positive interactions are activations and negative ones inhibitions [2, 3]. Yet, since the edge-weights and dynamic model associated to these networks are unknown, this provides “qualitative” information of the system only.

The classical *sign-stability criterion* introduced in the 70's opened the door to study the qualitative stability of linear systems, requiring to know only the sign-pattern of their interconnection network in order to conclude stability [4, 5]. Its impact was profound in diverse fields including ecology [6], economy [7], and more recently control [8] and network science [9]. However, the classical sign-stability criterion cannot be applied to nonlinear phenomena such as oscillations, bistability, chaos and so on that often appear in engineering, biological and economic systems [10]. For nonlinear systems, stability is preserved under cascade (i.e., series) interconnections under quite general conditions [11, 12]. Small-gain conditions have also been proposed to understand the stability of nonlinear systems with particular interconnection networks [13, 14], but these conditions are not completely qualitative as we need to establish the small-gain property of the nodal dynamics. Thus, a qualitative stability criterion for nonlinear systems beyond the “cascade of stable systems is stable” is lacking.

In this note, we use contraction theory [15] in order to extend the sign-stability criterion to nonlinear systems. Remarkably, almost the same conditions as in the linear sign-stability criterion imply stability in nonlinear systems. Indeed,

the conditions for sign-stability of linear and nonlinear systems are identical if the “asymmetries” of the system are constant. Otherwise it becomes necessary that the intrinsic stability of isolated nodes is sufficiently strong. The rest of the note is organized as follows. Section II presents the problem statement and our main results, which are proved in Section III. Section IV extends our nonlinear sign-stability criterion to delayed interconnections and modules (i.e., sets of nodes), and presents a discussion on the conditions for stability. Section V contains some concluding remarks.

II. PROBLEM STATEMENT AND MAIN RESULTS

Consider a system composed of n nodes where the scalar $x_i(t)$ denotes the state of node i at time t . The state of a node may represent the abundance of certain specie in an ecological system, or the expression level of some gene in a gene regulatory network, and so on. Suppose we are given a directed network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated to the system with vertex set $\mathcal{V} = \{x_1, \dots, x_n\}$ and edge set \mathcal{E} . An edge $x_j \rightarrow x_i$ in the network corresponds to a *direct* influence of x_j on x_i . We are also given the “sign” of each edge in the network, associating positive sign to “activation” and negative sign to “repression”.

Our objective is to characterize sufficient conditions for the stability of the system from the knowledge of \mathcal{G} . To address this problem, let us assume that the system dynamics satisfy the differential equations

$$\dot{x}_i = f_i(x_1, \dots, x_n, t), \quad i = 1, \dots, n, \quad (1)$$

for some smooth functions $f_i(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$, $x = (x_1, \dots, x_n)^T$. We might not know these functions exactly, but we assume we know some of their structure as precised later on.

Under model (1), the edge $x_j \rightarrow x_i$ exists in \mathcal{G} if and only if the function $a_{ij}(x, t) = \partial f_i(x, t) / \partial x_j$ is not identically zero. Similarly, this edge is positive if $a_{ij}(x, t) > 0$ and negative if $a_{ij}(x, t) < 0$ [3]. In general, the sign of $a_{ij}(x, t)$ changes with time but this will not introduce any conceptual difficulty to our analysis. In particular, when the functions $f_i(x, t)$ are time invariant and monotone in x —as in most models of gene regulation—the sign of $a_{ij}(x, t)$ remains constant.

We use the following notion of stability [15]:

Definition 1. *System (1) is contracting if for any two initial conditions their corresponding trajectories converge exponentially fast towards each other.*

A contracting system forgets exponentially fast its initial condition converging towards a unique trajectory. In contrast to Lyapunov stability, such limiting trajectory is not necessarily constant (i.e., an equilibrium). Historically, basic convergence results on contracting systems can be traced back to [16] using Finsler metrics, and also to [17, 18] and [19].

Our main result is:

Theorem 1. *System (1) is contracting provided that:*

- (i) *Reciprocate interactions have opposite signs: if $x_i \rightarrow x_j$ and $x_j \rightarrow x_i$ are edges in the network, $i \neq j$,*

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then $a_{ji}(x, t) = -b_{ij}(x, t)a_{ij}(x, t)$ for some function $b_{ij}(x, t) > 0$.

- (ii) Self-loops are negative and strong enough to overcome the logarithmic derivative of the asymmetries: there exists functions $\alpha_i(x, t) > 0$ such that $a_{ii}(x, t) = -\alpha_i(x, t)$ and

$$\alpha_i(x, t) > \frac{1}{2} \sum_{j \in \mathcal{N}_i} \frac{\dot{b}_{ij}(x, t)}{b_{ij}(x, t)}, \quad i = 1, \dots, n,$$

where \mathcal{N}_i is the set of feedback neighbors of node i .

- (iii) \mathcal{G} does not contain cycles of length 3 or more.

Corollary 1. Suppose that the asymmetries $b_{ij} > 0$ are constant (i.e., reciprocate interactions have the same functional form). Then (1) is contracting if $a_{ii}(x, t) < 0$ and \mathcal{G} does not have cycles of length 3 or more.

We say that (1) is *sign-stable* if it satisfies the conditions of Theorem 1. When these conditions hold in a region $\Omega \subseteq \mathbb{R}^n$ only, we say that (1) is sign-stable in Ω . The functions $b_{ij}(x, t)$ in condition (i) represent the *asymmetry* between the edges involved in a negative feedback loop, Fig.1a. Corollary 1 implies that the sign-stability conditions for linear and nonlinear systems are identical if the asymmetries in the system are constant. When the asymmetries are not constant, condition (ii) requires that the intrinsic stability of the nodes (represented by their contraction rates $\alpha_i(x, t) > 0$) dominates their logarithmic derivative. This condition of sufficiently large contraction rates of the isolated nodes turns out to be necessary for contraction with time-varying asymmetries, and it can also be stated in terms of how “fast” the asymmetry needs to be (Section IV-B). The *feedback neighbors* \mathcal{N}_i of node x_i is the set of nodes x_j such that the edges $x_i \rightarrow x_j$ and $x_j \rightarrow x_i$ exist. A *cycle* in \mathcal{G} is a sequence of edges (excluding self loops) that starts and ends in the same node, and its length is the number of edges it contains. For example $\{x_i \rightarrow x_j, x_j \rightarrow x_k, x_k \rightarrow x_i\}$ with $i \neq j \neq k$ is a cycle of length 3. Condition (iii) is related to the fact that systems with cycles of length 3 require specific tuning of their edge-weights to be stable. In the case of linear systems with $n = 3$, this can be seen from the root-locus diagram: there are three branches and, by symmetry, at least one branch crosses to the right half-plane of the complex plane. Consequently, the stability depends on the edge-weights of the network and we cannot determine stability based on its signs only. In Section IV we extend Theorem 1 to time-delayed interconnections, and to interconnections of modules instead of nodes. When the functions f_i are linear and time invariant, our result reduces to the classical sign-stability criterion [4, 5].

Example 1: Applying Corollary 1, the system $\dot{x}_1 = -x_1 - x_1x_2$, $\dot{x}_2 = x_1^2 - x_2 - x_2x_3$, $\dot{x}_3 = x_2^2 - x_3$ is sign-stable in the region $\Omega = \{x \in \mathbb{R}^3 | x_2 > -1, x_3 > -1\}$. Indeed, it satisfies condition (i) with $b_{12} = b_{23} = 2$, and condition (ii) because $a_{11} = -1 - x_2 < 0$, $a_{22} = -1 - x_3 < 0$, $a_{33} = -1$. Condition (iii) is satisfied because its associated network does not have cycles of length 3 or more, Fig.1b. Furthermore, the origin $x \equiv 0$ is exponentially stable since it is a trajectory in Ω .

Our proof of the main result is based on two observations. First, the network of a sign-stable system is composed of cascades of what we call “feedback chains” (Proposition 2).

Feedback chains are systems recursively built using negative feedback interconnections under the constraint that they do not have cycles of length 3 or more. Second, feedback chains are contracting provided that their nodes, when isolated, have a large enough contraction rate to dominate the logarithmic derivative of the asymmetry of the system (Proposition 1). From these two observations, the proof of Theorem 1 follows directly, because the cascade interconnection of contracting systems remains contracting [15].

III. PROOF OF THE MAIN RESULT

A. Contraction theory, and its tools to prove stability.

Let $f = (f_1, \dots, f_n)^\top$ allowing us to rewrite (1) as $\dot{x} = f(x, t)$. Denote its Jacobian by $A(x, t) = \frac{\partial f(x, t)}{\partial x}$, and note its (i, j) entry is the function $a_{ij}(x, t)$ used earlier to construct the network \mathcal{G} . Hereafter, we often omit the arguments of the functions to improve readability unless they are relevant for the discussion. Contraction theory uses the fact that exponential stability of the *differential system* $\dot{\delta}_x(t) = A(x, t)\delta_x(t)$ implies that system (1) is contracting [15]. In order to prove contraction, a necessary and sufficient condition is a symmetric positive-definite matrix $M(t, x)$ such that

$$L(A, M) = \dot{M} + A^\top M + MA \prec 0. \quad (2)$$

The matrix M introduces a metric in the differential coordinates, and we say that the system is *diagonally contracting* if M in (2) can be taken diagonal. Changes of coordinates are also useful, as illustrated in Example 4.

B. Feedback chains, and their diagonal contraction properties.

The basic building block for our analysis will be the negative feedback interconnection for $n = 2$ nodes shown in Fig.1a. Its corresponding Jacobian is

$$A_{12} = \begin{pmatrix} -\alpha_1 & a_{12} \\ -b_{12}a_{12} & -\alpha_2 \end{pmatrix}, \quad (3)$$

with $\alpha_i(x, t) > 0$ and $b_{12}(x, t) > 0$. The particular structure of (3) suggests using the diagonal metric $D_{12} = \text{diag}\{b_{12}, 1\}$ to prove contraction:

$$L(A_{12}, D_{12}) = \begin{pmatrix} -2\alpha_1 b_{12} + \dot{b}_{12} & 0 \\ 0 & -2\alpha_2 \end{pmatrix}$$

which will be uniformly negative-definite if

$$\alpha_1 > 0.5 \dot{b}_{12}/b_{12}, \quad \alpha_2 > 0.$$

Hence a negative feedback interconnection between two nodes is contracting provided that condition (ii) holds. In this case not only the metric D_{12} is diagonal, but also $L(A_{12}, D_{12})$ is diagonal. This property will be instrumental in order to extend this result to more general feedback interconnections, which we call “feedback chains”. For linear systems, the diagonal stability of systems with the structure (3) is consequence of the so-called Schwarz form [20]. Here contraction theory naturally extends this property to nonlinear systems.

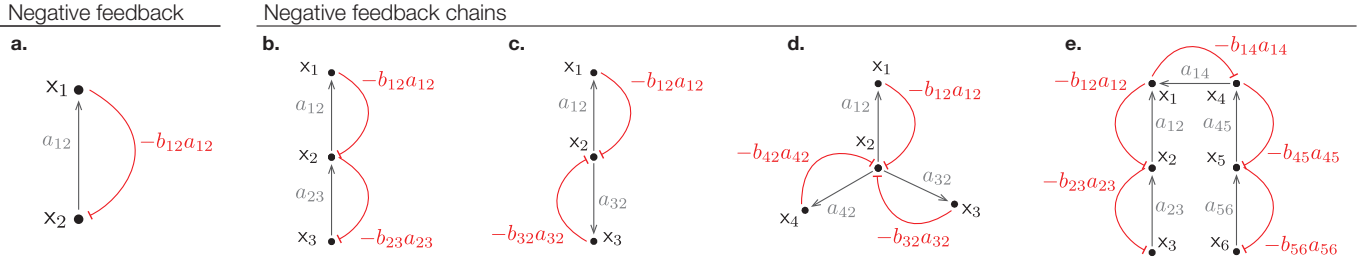


Fig. 1. **Two-node negative feedback interconnection, and some examples of negative feedback chains.** In all networks shown, the sign of a_{ij} may change with time because it is a function of $x(t)$ and t . Hence, the different edge representation for pairs of activation “ \rightarrow ” and inhibition “ \dashv ” indicates that a_{ij} and $b_{ij}a_{ij}$ have opposite signs at each time instant. All nodes have negative self-loops $a_{ii} = -\alpha_i < 0$, which are omitted when a network is displayed in order to improve readability.

The systems shown from Fig.1c to Fig.1e are other examples of negative feedback interconnections. Their linear versions are no longer Schwarz forms of any order, so we call them *feedback chains*. Feedback chains are recursively built starting from a negative feedback interconnection between two nodes, and adding new nodes by interconnecting each one of them to a single existing node using negative feedback (i.e., reciprocate interactions with $b_{ij} > 0$). By their construction, feedback chains do not contain cycles of length 3 or more.

Example 2: Consider Fig.1c and let A_{123} denote its Jacobian

$$A_{123} = \begin{pmatrix} -\alpha_1 & a_{12} & 0 \\ -b_{12}a_{12} & -\alpha_2 & -b_{32}a_{32} \\ 0 & a_{32} & -\alpha_3 \end{pmatrix}.$$

If condition (ii) holds, the first cycle $\{x_1, x_2\}$ is contracting with metric $D_{12} = \text{diag}\{b_{12}, 1\}$; similarly, the second one $\{x_2, x_3\}$ is contracting with metric $D_{23} = \text{diag}\{1, b_{32}\}$. We naturally combine both metrics $D_{123} = \text{diag}\{b_{12}, 1, b_{32}\}$ obtaining

$$L(A_{123}, D_{123}) = \begin{pmatrix} L(A_{12}, D_{12}) & 0_{2 \times 1} \\ 0_{1 \times 2} & -2\alpha_3 b_{32} + \dot{b}_{32} \end{pmatrix}$$

proving that the system corresponding to Fig.1c is contracting provided that node x_3 satisfies condition (ii). Here $L(A_{123}, D_{123})$ is again diagonal, and recursively depends on $L(A_{12}, D_{12})$. The system in Fig. 1b. can be similarly analyzed defining $a_{32} = -b_{23}a_{23}$ and $b_{32} = b_{23}^{-1}$.

Now consider network Fig.1d obtained by adding the node x_4 to the feedback chain of Fig.1c. For subsystem A_{123} we have the metric D_{123} , and for the subsystem A_{24} we have the metric $D_{24} = \text{diag}\{1, b_{42}\}$. Naturally combining both metrics $D_{1234} = \text{diag}\{b_{12}, 1, b_{32}, b_{42}\}$ we obtain

$$L(A_{1234}, D_{1234}) = \begin{pmatrix} L(A_{123}, D_{123}) & 0_{3 \times 1} \\ 0_{1 \times 3} & -2\alpha_4 b_{42} + \dot{b}_{42} \end{pmatrix}$$

which is again diagonal and proves that the system corresponding to network Fig.1d is contracting if it satisfies condition (ii).

The above example show how to recursively build diagonal metrics to prove contraction of feedback chains, illustrating the following result.

Proposition 1. *Provided that nodes satisfy condition (ii), a negative feedback chain is diagonally contracting.*

Proof: We prove the claim by induction on the dimension of the chain. Let Σ_i denote a feedback chain of dimension i

and A_i its Jacobian. We have shown that Σ_2 (i.e., two-node negative feedback) is diagonally contracting: there exists a diagonal metric D_2 such that $L(A_2, D_2) \prec 0$. Furthermore, $L(A_2, D_2)$ is also diagonal.

Now suppose that Σ_i is diagonally contracting and $L(A_i, D_i) \prec 0$ is diagonal. Let x_{i+1} be the new added node. By relabeling the nodes in Σ_i , we assume that x_{i+1} will be connected to x_i without loss of generality. Therefore the Jacobian of Σ_{i+1} is

$$A_{i+1} = \begin{pmatrix} A_i & -b_{i+1}a_{i+1}e_i \\ a_{i+1}e_i^\top & -\alpha_{i+1} \end{pmatrix} \quad (4)$$

where $e_i = (0, \dots, 0, 1) \in \mathbb{R}^i$. Here we have used a_{i+1} and b_{i+1} instead of $a_{i+1,i}$ and $b_{i+1,i}$ to simplify the notation. As noted in Example 2, we may rewrite the Jacobian using $a_{i,i+1} = -b_{i+1,i}a_{i+1,i}$ and $b_{i,i+1} = b_{i+1,i}^{-1}$ so the expression (4) can be considered without loss of generality. Based on $D_i \in \mathbb{R}^{i \times i}$, we build the new metric

$$D_{i+1} = \begin{pmatrix} D_i & 0 \\ 0 & d_{ii}b_{i+1} \end{pmatrix}$$

where $d_{ii} > 0$ is the i -th diagonal element of D_i . With this choice we obtain

$$D_{i+1}A_{i+1} = \begin{pmatrix} D_iA_i & -b_{i+1}a_{i+1}D_ie_i \\ b_{i+1}a_{i+1}d_{ii}e_i^\top & -d_{ii}b_{i+1}\alpha_{i+1} \end{pmatrix}.$$

Notice that $D_ie_i = d_{ii}e_i$, so we get

$$L(A_{i+1}, D_{i+1}) = \begin{pmatrix} L(A_i, D_i) & 0 \\ 0 & -2\alpha_{i+1}d_{ii}b_{i+1} + \frac{d}{dt}\{d_{ii}b_{i+1}\} \end{pmatrix}.$$

From the induction hypothesis, we know that $L(A_i, D_i)$ is diagonal and uniformly negative definite. On the other hand, notice that $d_{ii} = \prod_{j \in \mathcal{N}_i} b_j$ where \mathcal{N}_i is the set feedback neighbors of node x_i discarding x_{i+1} . Using these two facts and the rule for the derivative of product of functions, $L(A_{i+1}, D_{i+1}) \prec 0$ provided that

$$-2\alpha_{i+1}b_{i+1} \left(\prod_{j \in \mathcal{N}_i} b_j \right) + b_{i+1} \left(\prod_{j \in \mathcal{N}_i} b_j \right) \left(\frac{\dot{b}_{i+1}}{b_{i+1}} + \sum_{j \in \mathcal{N}_i} \frac{\dot{b}_j}{b_j} \right) < 0$$

which is condition (ii) once it is divided by $\prod_{j \in \mathcal{N}_i} b_j$. ■

Despite it is possible to prove Proposition 1 without using recursion, the above proof is more natural from the point of view of a growing network, building larger metrics based on existing smaller ones. This recursive method will also be useful to extend the sign-stability criterion to delayed interconnections and modules (Section IV).

C. The network structure of sign-stable systems.

To conclude the proof of the main result, we show that when \mathcal{G} does not contain cycles of length 3 or more—that is, condition (iii) holds—then the network is composed of cascades of feedback chains. We start showing that feedback chains are the minimal building blocks of networks with such constraint, and then we characterize how to interconnect feedback chains while keeping this constraint.

Lemma 1. *Adding a new edge to a feedback chain produces a cycle of length 3 or more.*

Proof: By construction, a feedback chain is strongly connected (i.e., there is a direct path between any two nodes) and there is no cycle of length 3 or more. Now suppose we add a new edge $x_j \rightarrow x_i$. Yet, there is already a path in the network from x_i to x_j and since the edge is new, the length of this path should be 2 at least. Hence the cycle obtained by the new edge and the already present path has length 3 or more. ■

Lemma 2. *Two feedback chains can be interconnected without creating a cycle of length 3 or more only by (i) a two-node feedback (so both feedback chains are merged into a larger one), or (ii) in cascade.*

Proof: (i) If we interconnect two chains by a feedback we create a larger feedback chain, see Fig.2a. Lemma 1 shows that adding another edge to this network creates a cycle of length 3 or more, Fig.2b. (ii) Consider an edge going from chain Σ_a to chain Σ_b and suppose there is another edge going from Σ_b to Σ_a . This creates a cycle of length 3 or more, because each feedback chain is strongly connected (see Fig.2d). Consequently, edges can only go from one feedback chain to the other, Fig.2c. ■

Proposition 2. *Under conditions (i) and (iii), the network \mathcal{G} of the system is a cascade of negative feedback chains.*

The proof of Proposition 2 is a direct consequence of Lemma 1 and 2. Figure 2a-2d illustrates these lemmas. Together with Proposition 1, Proposition 2 completes the proof of Theorem 1 because the cascade of contracting systems is contracting [15]. Corollary 1 follows from Theorem 1 because if b_{ij} is constant then its logarithmic derivative is zero.

IV. DISCUSSION AND EXTENSIONS

A. Delayed interconnections.

Delays in interconnections happen in natural and technological systems. Here we remark that for systems with uniformly bounded interactions, there exists a critical threshold

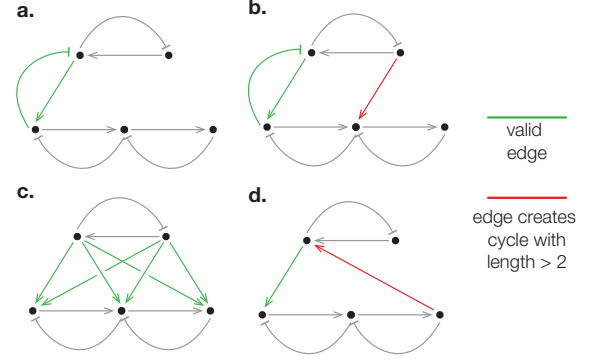


Fig. 2. **Interconnection of feedback chains without creating cycles of length > 2 .** **a.** Interconnecting feedback chains using two-node feedback does not create cycles of length > 2 , merging together two chains into a larger one (Lemma 2-i). **b.** Adding any edge to a feedback chain always create cycles of length > 2 (Lemma 1). **c.** Interconnecting two feedback chains in cascade does not create cycles of length > 2 (Lemma 2-ii). **d.** If the interconnection between two feedback chains have different directions and are not feedback to the same node, they necessarily create a cycle with length > 2 (Lemma 2-ii).

for the contraction rates of the nodes such that the system is contracting for *any* value of delay.

Example 3: Consider a two-node negative feedback interconnection with delays $T_1, T_2 \geq 0$, which corresponds to the following differential system

$$\begin{aligned}\dot{\delta}_{x_1}(t) &= -\alpha_1 \delta_{x_1}(t) + a_{12} \delta_{x_2}(t - T_2), \\ \dot{\delta}_{x_2}(t) &= -b_{12} a_{12} \delta_{x_1}(t - T_1) - \alpha_2 \delta_{x_2}(t).\end{aligned}$$

Suppose there exists a constant $\Gamma \geq 0$ such that $|a_{12}(x, t)| b_{12}(x, t) \leq \Gamma, \forall (x, t)$. Consider $V(\delta_x) = \delta_x^T D_{12} \delta_x$ with $D_{12} = \text{diag}\{b_{12}, 1\}$ as previously chosen. In contrast to our previous analysis, the crossed terms do not cancel due to the delays

$$\begin{aligned}\dot{V} &= \delta_x^T(t) L(A_{12}, D_{12}) \delta_x(t) + \\ & 2a_{12}b_{12} [\delta_{x_1}(t) \delta_{x_2}(t - T_2) - \delta_{x_1}(t - T_1) \delta_{x_2}(t)],\end{aligned}$$

but note that the first term of the above equation is still negative. Using the upper bound for the interactions Γ and the inequality

$$2|\delta_{x_i}(t) \delta_{x_j}(t - T_j)| \leq \delta_{x_i}^2(t) + \delta_{x_j}^2(t - T_j) \quad (5)$$

we obtain $\dot{V} \leq \delta_x^T(t) L(A_{12}, D_{12}) \delta_x(t) + \Gamma [\delta_{x_1}^2(t) + \delta_{x_2}^2(t - T_2) + \delta_{x_1}^2(t - T_1) + \delta_{x_2}^2(t)]$. Consider now the strictly positive function

$$V_{12} = \int_{t-T_1}^t \delta_{x_1}^2(\sigma) d\sigma + \int_{t-T_2}^t \delta_{x_2}^2(\sigma) d\sigma$$

with time derivative $\dot{V}_{12} = \delta_{x_1}^2(t) - \delta_{x_1}^2(t - T_1) + \delta_{x_2}^2(t) - \delta_{x_2}^2(t - T_2)$. When we combine these two functions $V_{tot} = V + \Gamma V_{12}$ the delayed terms cancel out:

$$\dot{V}_{tot} \leq -(2\alpha_1 b_{12} - \dot{b}_{12} - 2\Gamma) \delta_{x_1}^2(t) - 2(\alpha_2 - \Gamma) \delta_{x_2}^2(t).$$

Consequently, if the contraction rates of the isolated nodes satisfy

$$\alpha_1 > (0.5 \dot{b}_{12} + \Gamma) / b_{12}, \quad \alpha_2 > \Gamma, \quad (6)$$

then the system is contracting. In general, the bound Γ may increase with the state. However, when the interconnection

functions are independent of the state (e.g., time varying functions), condition (6) is sufficient to ensure contraction for *any* value of the delays. Indeed, in the case of linear time-invariant systems, the characteristic equation of the system is

$$1 + L(s) = 1 + \frac{a_{12}b_{12}e^{-s(T_1+T_2)}}{(s + \alpha_1)(s + \alpha_2)}$$

where $s \in \mathbb{C}$ is the Laplace variable. Observing that for any given $\Gamma \geq 0$ there exists constant $\alpha_1, \alpha_2 > 0$ such that $|L(i\omega)| < 1$ for all $T_1, T_2 \geq 0$, the Nyquist criterion immediately implies that the system is stable for *any* delay.

The method of the above example applies directly to the general case. We start with $V = \delta_x^\top D \delta_x$ with the metric D constructed from Proposition 1. Next we apply (5) to replace the cross terms by sums of squares. Finally we add $V_{i:k} = \sum_{j=i}^k \int_{t-T_j}^t \delta_{x_j}^2(\sigma) d\sigma$ to replace the delayed squared terms by squared terms without delays. Note this is not restricted to the particular network structure of sign-stable systems.

B. Time-varying asymmetries need large enough contraction rates.

The linear sign-stability criterion [4, 5] and the nonlinear criterion with constant asymmetries (Corollary 1) do not require the “large enough” contraction rate condition for α_i —condition (ii) of Theorem 1— which may be difficult to establish in practice. In this sense, both criteria are purely qualitative (despite its necessary to establish linearity or constant asymmetries in advance). Nevertheless, even in the linear case, the condition of large enough contraction rates turns out to be necessary when the asymmetries are time-varying. This condition can also be expressed in terms of how “fast” the asymmetries need to be. We illustrate these two statements using the following example.

Example 4: Consider a linear time-varying instance of the two-node negative feedback interconnection (3), using $\alpha_1 = \alpha_2 = \alpha = \text{const.}$, $a_{12} = 1$ and $b_{12} = b(t) = 1 + 0.9 \sin(t) > 0$. Using numerical simulations, we conclude this system is unstable for $\alpha \leq 0.05$ but stable for $\alpha \geq 0.09$, see Fig. 3. In other words, it is necessary a large enough contraction rate $\alpha > 0$ for stability. Next recall from [21, Theorem 2] that the linear system $\dot{\delta}_x(t) = A(\omega t)\delta_x(t)$ is globally exponentially stable $\forall \omega > \omega^*$ for some finite $\omega^* > 0$ if there exists $T > 0$ such that the “average” $\int_t^{t+T} A(\tau) d\tau$ is uniformly negative definite for all $t \geq 0$. In order to apply this result, we first rewrite our example in coordinates $\Theta(t) = \text{diag}\{1, b(t)^{-1/2}\}$. In these coordinates the Jacobian reads $F = \Theta A \Theta^{-1} + \dot{\Theta} \Theta^{-1}$, so we obtain $F(t) = \text{diag}\{-2\alpha, -2\alpha + \dot{b}(t)/b(t)\}$. Now we can compute

$$\int_t^{t+T} F(\tau) d\tau = \begin{pmatrix} -2\alpha T & 0 \\ 0 & -2\alpha T + \int_t^{t+T} \frac{d}{dt} \ln(b) d\tau \end{pmatrix}.$$

The above matrix is negative definite in the identity metric if

$$-2\alpha T + \ln \frac{b(t+T)}{b(t)} < 0$$

which can be rewritten as $b(t+T) < e^{2\alpha T} b(t)$. For any $\alpha > 0$ there is always $T > 0$ satisfying this condition if

the function $b_{12}(t)$ is uniformly bounded from above and below. Indeed, if $b_{12}(t) \geq \epsilon > 0$ and $b_{12}(t) \leq \gamma$ we can choose any $T > (1/2\alpha) \ln(\gamma/\epsilon)$. This shows that for any $\alpha > 0$, the system is contracting if ω is large enough. In other words, if the asymmetry is bounded from above and below and fast enough. In our example with $b(\omega t) = 1 + 0.9 \sin(\omega t)$ and $\alpha = 0.05$, the system becomes stable by increasing the frequency to $\omega = 1.1$.

On the other hand, Theorem 1 also implies that for any given $\alpha > 0$ there exists $\omega_* > 0$ small enough such that the system remains contracting for any $\omega < \omega_*$. Indeed, our example with $\alpha = 0.05$ becomes stable by decreasing the frequency of $b(\omega t)$ to $\omega = 0.5$.

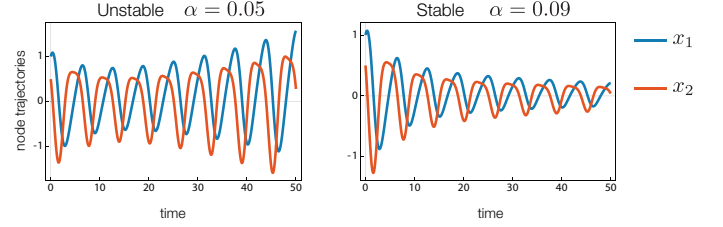


Fig. 3. **For systems with time-varying asymmetry, a large enough contraction rate is necessary.** Trajectories of the system of Example 3 with initial condition $x(0) = (1, 0.5)^\top$. We observe that the system is unstable if the contraction rates of the nodes $\alpha \leq 0.05$, but stable if $\alpha \geq 0.09$.

By regarding the differential system $\dot{\delta}_x = A(x(t), t)\delta_x$ as a linear time-varying system and using the recursive proof of Proposition 1, it is straightforward to extend the analysis of the above example to feedback chains and hence to sign-stable systems. Consequently, time-varying asymmetries do not detriment stability if they are either slow enough or fast enough compared to the system dynamics. Biological and man-made systems often have time scale separations, so we can exploit them to establish the sign-stability of networked systems [22, 23]. With the help of singular perturbation techniques, time scale separations in the nodal dynamics can be used to apply the sign-stability criterion to slow nodes only. Importantly, Example 4 implies that a pure qualitative criterion for the stability of systems with time-varying asymmetries is impossible.

C. Vector nodal dynamics.

Consider the negative feedback interconnection between two modules with states $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, corresponding to the following differential system

$$\dot{\delta}_{x_1} = A_{11}\delta_{x_1} + A_{12}\delta_{x_2}, \quad \dot{\delta}_{x_2} = -b_{12}A_{12}^\top\delta_{x_1} + A_{22}\delta_{x_2},$$

with $b_{12}(x, t) > 0$ scalar and $A_{ij}(x, t) = \partial f_i(x, t)/\partial x_j \in \mathbb{R}^{n_i \times n_j}$. Suppose that, when isolated, each module is contracting with metric M_i and that these metrics satisfy the compatibility condition $M_1 A_{12} = A_{12} M_2$. Using the metric $M = \text{blkdiag}\{b_{12}M_1, M_2\}$ we obtain

$$L(A, M) = \begin{pmatrix} b_{12}[L(A_{11}, M_1) + (\dot{b}_{12}/b_{12})M_1] & 0 \\ 0 & L(A_{22}, M_2) \end{pmatrix}$$

and the system remains contracting if $L(A_{11}, M_1) + (\dot{b}_{12}/b_{12})M_1 \prec 0$ and $L(A_{22}, M_2) \prec 0$, in complete analogy to the scalar case of Section III-B. Consequently, the sign-stability criterion can be straightforwardly extended to modules by extending Proposition 1, first modifying condition (ii) as follows

$$L(A_{ii}, M_i) + \sum_{j \in \mathcal{N}_i} \frac{\dot{b}_{ij}}{b_{ij}} M_i \prec 0, \quad i = 1, \dots, n;$$

and second requiring the additional *metric compatibility* condition $M_i A_{ij} = A_{ij} M_j, \forall j \in \mathcal{N}_i$. Without this compatibility condition, the conditions that each module needs to satisfy in order to guarantee stability of the whole network are not local anymore (i.e., do not depend on the module's feedback neighbors only) because the block-diagonal structure is lost.

V. CONCLUDING REMARKS

This paper bridges a theoretical gap by extending the sign-stability criterion to nonlinear systems, providing also extensions to consider modules and delayed interconnections. In practice, it is rare that large networked systems are entirely sign-stable. Rather, the nonlinear sign-stability criterion can be used to identify portions of the system (i.e., modules) that are stable by the “topological” design of their interconnection network [9]. It is also possible to combine the sign-stability criterion with other interconnections that preserve contraction, such as “centralized” ones [24]. From an engineering viewpoint, our results allow to recursively build large networks which automatically preserve stability using simple conditions on times scales, delays and signs of the interconnections.

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